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Classification on the Average of Random Walks

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During the last decade many attempts have been made to characterize absence of spontaneous breaking of continuous symmetry for the Heisenberg model on graphs by using suitable classifications of random walks (refs. 4 and 10). We propose and study a new type problem for random walks on graphs, which is particularly interesting for disordered graphs. We compare this classification with the classical one and with an analogous one introduced in ref. 4. Various examples, that are not space-homogeneous, are provided.

KEY WORDS: Random walk; limit on the average; generating function; summability methods.

1. INTRODUCTION

Random walks on graphs provide a mathematical model in many scientific areas, from finance (financial modelling), to physics (magnetization properties of metals, evolution of gases), and biology (neural networks, disease spreading). In particular graphs describe the microscopical structure of solids, ranging from very regular structures like crystals or ferromagnetic metals which are viewed as Euclidean lattices, to the irregular structure of glasses, polymers, or biological objects.

Geometrical and physical properties of these discrete structures are linked by random walks (especially the simple random walk), which usually describe the diffusion of a particle in these more or less regular media.

An interesting feature of random walks on graphs is their large time scale asymptotics which is deeply connected with the concept of recurrent or transient random walk. This classification was first introduced by Pólya[(15)]

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for simple random walks on lattices to distinguish between random walks which return to the starting point with probability one (these are recurrent), and those whose return probability is less than one (which are transient).

We observe that in a vertex-transitive graph (such as the lattice $\mathbb{Z}^d$) the return probabilities of the simple random walk do not depend on the starting vertex; but in the case of a general irreducible random walk they may differ from vertex to vertex, although being strictly less than one in one vertex is equivalent to being strictly less than one in any vertex. The distinction between recurrent and transient random walks is known as the type problem (for the type problem for random walks on infinite graphs, see Woess[18]).

It has been recently observed that even though the type of a random walk describes local properties of the physical model, average values of return probabilities over all starting sites play a key role in the comprehension of the macroscopical behaviour of the model itself (like spontaneous breaking of continuous symmetries,[5] critical exponents of the spherical model,[6] or harmonic vibrational spectra[2]). In particular Merkl and Wagner[10] have shown that there is no spontaneous breaking of continuous symmetries (for certain spin models) on recurrent graphs, but since recurrence is not necessary, maybe a different notion of recurrence could characterize the phenomenon.

As a conjectured characterization Burioni *et al.*[4] proposed recurrence on the average, defining a new type problem: the type problem on the average.

Definition 1.1. Given a random walk adapted to a graph X, its family of generating functions of the n-step first time return probabilities $\{F(x, x \mid z)\}_{x \in X}$, and a reference vertex $o \in X$ the random walk is transient on the average ($\mathrm{TOA_t}$) if

$$\lim_{z \to 1^-} \lim_{n \to \infty} \frac{\sum_{x \in B(o,n)} F(x, x \mid z)}{|B(o, n)|} < 1, \tag{1}$$

and recurrent on the average ($\mathrm{ROA_t}$) if the value of the limit is equal to 1 ($B(o, n)$ is the closed ball, with respect to the natural metric of X, of center o, and radius n, $|\cdot|$ denotes cardinality).

The "average" mentioned in the name given to this new type problem is a repeated average over balls with fixed center and increasing radii (of course existence of the limit of these averages is implicitly required). This procedure is a particular case of the following: given a sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ of probability measures on the set X, for each n we consider the expected value of $F(\cdot, \cdot \mid z)$ with respect to λ_n, we take the limit of these averages when n goes to infinity and we evaluate the limit for z going to 1 (to fit definition (1) just take $\lambda_n(x) = \mathbb{1}_{B(o,n)}(x)/|B(o,n)|$).

From a mathematical point of view the definition of this "limit on the average" leads to some problems, like the existence of the limit, the possibility of exchanging the order of the two limits and the dependence on the reference vertex o.

We provide an example of random walk which has no classification on the average in the above sense (Example 3.1). Thus the classification on the average is not complete, while the classical one in recurrent and transient random walks is complete (we will refer to the usual classification as the "local" one, in contrast with the one "on the average").

In order to overcome the technical problems associated to Definition 1.1 we propose a more general approach (averages are taken with respect to general sequences $\lambda = \{\lambda_n\}_{n \in \mathbb{N}}$, although a special attention is paid to the example of averages over balls) and a different definition of classification on the average (Definition 2.6: transient and recurrent on average walks are denoted by λ-TOA and λ-ROA). This classification is complete and is in many cases an extension of Definition 1.1.

We compare the two definitions and we give some criteria (Section 3.1). In particular it appears that the two classifications coincide when the generating function of the first time return probabilities is a totally convergent series (Theorem 3.2(ii)): sufficient conditions for this property to be satisfied are provided in Proposition 3.3. Other conditions are provided by suitable uniform upper estimates for the n-step transition probabilities $p^{(n)}(x, x)$; such estimates can be found in many papers: see, for instance, Barlow *et al.*,[1] Coulhon and Grigory'an,[7] Grigor'yan and Telcs,[8, 9] and Telcs.[17]

The question of the independence of the classification on the reference vertex o (and in general on the choice of the sequence $\{\lambda_n\}_{n \in \mathbb{N}}$) is investigated in Section 3.2.

In Section 3.3 we compare the classical ("local") classification and the classification on the average. In particular a flow criterion on the average is proven (Theorem 3.13), which should be compared with the "classical" one of Lyons[14] and Yamasaki.[19] Throughout Section 3 various examples are provided to clarify the relations between the classifications (see Table I for a complete overview).

In Section 4 we connect the behaviour of the random walk on the subgraph to the behaviour of the random walk on the whole graph. Section 5 discusses some open problems. The appendix shows that the family of "measurable" sets, in the sense that we define in Section 2, is not an algebra.

Table I. Comparison between the Three Classifications

	λ-TOA, λ-TOA$_t$	λ-TOA, λ-ROA$_t$	λ-ROA, λ-TOA$_t$	λ-ROA, λ-ROA$_t$
Recurrent	impossible (Remark 3.8)	impossible (Theorem 3.2(i))	Example 3.12	$\mathbb{Z}^2$
Transient	$\mathbb{Z}^3$	impossible (Theorem 3.2(i))	Example 3.4	Example 3.9

2. BASIC DEFINITIONS

2.1. Random Walks

We recall here some notation. Given a random walk (X, P) we denote by $p^{(n)}(x, y)$ the n-step transition probabilities from x to y ($n \geqslant 0$) and by $f^{(n)}(x, y)$ the probability that the random walk starting from x hits y for the first time after n steps ($n \geqslant 1$). Then we define the corresponding generating functions $G(x, x \mid z) = \sum_{n \geqslant 0} p^{(n)}(x, x) z^n$ (the Green function) and $F(x, x \mid z) = \sum_{n \geqslant 1} f^{(n)}(x, x) z^n$, where $x \in X$, $z \in \mathbb{C}$ (further details can be found in Woess,[18] Chapter I.1.B, where F is called U). It is usual to write $F(z)$ instead of $F(\cdot, \cdot \mid z)$ and F (or $F(\cdot, \cdot)$) instead of $F(\cdot, \cdot \mid 1)$.

An irreducible random walk (X, P) is recurrent if $F(x, x)$ for some $x \in X$ (equivalently for all x) and transient if $F(x, x) < 1$ for some $x \in X$ (equivalently for all x).

We recall here the flow criterion which characterizes transient networks. One can associate an electric network to a reversible random walk (X, P) with reversibility measure m (see Woess,[18] Chapter I.2.A for the definition) in the following way. We endow any edge with an orientation $e = (e^-, e^+)$ and with a resistance $r(e) = (m(e^-)\, p(e^-, e^+))^{-1}$ (in the case of the simple random walk $r(e) = 1$ for every edge e).

A flow u from a vertex x to infinity with input i_0 at x is a function defined on $E(X)$ such that

$$\sum_{e:\, e^- = y} u(e) = \sum_{e:\, e^+ = y} u(e) + i_0 \delta_x(y), \qquad \forall y \in X.$$

The energy of u is defined as $\langle u, u \rangle := \sum_{e \in E(X)} u(e)^2\, r(e)$. The existence of finite energy flows is related with transience by the following theorem (see Theorem 2.12 in ref. 18).

Theorem 2.1. Let (X, P) be a reversible random walk. The following are equivalent:

(a) the random walk is (locally) transient;

(b) there exists $x \in X$ such that (equivalently for all $x \in X$) it is possible to find a finite energy flow with non-zero input, from x to infinity.

2.2. Limits on the Average

We define a large scale average associated to a sequence of probability measures on X.

Definition 2.2. Given a sequence $\lambda = \{\lambda_n\}_{n \in \mathbb{N}}$ of probability measures on X and a function $f: X \to \mathbb{R} \cup \{\pm\infty\}$ such that for any $n \in \mathbb{N}$, at least one of the functions $f^+ := \max(f, 0)$ and $f^- := -\min(f, 0)$ is λ_n-summable. We denote by *upper* and *lower limit on the λ-average* of f respectively

$$\bar{L}_\lambda(f) := \limsup_{n \to +\infty} \sum_{x \in X} f(x)\, \lambda_n(x),$$

$$\underline{L}_\lambda(f) := \liminf_{n \to +\infty} \sum_{x \in X} f(x)\, \lambda_n(x).$$

If $\bar{L}_\lambda(f) = \underline{L}_\lambda(f)$ we define

$$L_\lambda(f) = \lim_{n \to +\infty} \sum_{x \in X} f(x)\, \lambda_n(x).$$

Examples of λ (which we keep in mind as reference) are $\lambda_n(x) = \mathbb{1}_{B_n}(x)/|B_n|$ where either $B_n = B(o, n)$, or, more generally, $\{B_n\}_{n \in \mathbb{N}}$ is an increasing family of finite subsets whose union is X. In the first case (averaging over balls), we denote the map L_λ by L_o (analogously $\bar{L}_o$ and $\underline{L}_o$).

The limits on the average are particular cases of *summability methods* (see, for instance, ref. 13, Paragraph 4.10); if $\lim_{n \to \infty} \lambda_n(x) = 0$ for any $x \in X$ (i.e., every finite subset of X is measurable and its measure is zero), the limit on the average is called *regular*.

We note that the map L_λ may be defined on complex-valued functions as well, provided that the limit exists (indeed we will use the limit on the average of generating functions).

Definition 2.3. Let $\mathscr{D}(L_\lambda) := \{f: X \to \mathbb{C}: f \in \bigcap_n L^1(\lambda_n)$ and $L_\lambda(f)$ exists$\}$. If $A \subseteq X$ is such that $\mathbb{1}_A \in \mathscr{D}(L_\lambda)$, then A is L_λ-measurable (or briefly measurable) and with a slight abuse of notation, we write $L_\lambda(A)$

instead of $L_\lambda(\mathbb{1}_A)$ (and we call it the L_λ-measure of A or simply the measure of A).

From now on, whenever we write $L_\lambda(f)$ or $L_\lambda(A)$, we implicitly assume that the defining limits exist.

Remark 2.4. The set of L_λ-measurable subsets is not, in general, a σ-algebra (nor an algebra: see Proposition A.1). Anyway it is easy to show that: (i) if $L_\lambda(S)=0$ and $S' \subseteq S$ then $L_\lambda(S')=0$; (ii) if $L_\lambda(S)=0$ then for every bounded function f, we have that $\mathbb{1}_S f \in \mathscr{D}(L_\lambda)$ and $L_\lambda(\mathbb{1}_S f)=0$; (iii) $\bar{L}(S)+\underline{L}(S^c)=1$, for all $S \subseteq X$.

We restate the definition of the type problem according to ref. 4 (and we call it "thermodynamical" to distinguish it from the definition which we will give in a moment).

Definition 2.5. Let (X, P) be a random walk, and $\lambda=\{\lambda_n\}_{n\in\mathbb{N}}$ a sequence of probability measures on X. Suppose that $F(\cdot,\cdot\,|\,z) \in \mathscr{D}(L_\lambda)$, for all $z \in (\varepsilon, 1)$, for some $\varepsilon \in (0,1)$. The random walk is *thermodynamically transient on the average with respect to L_λ* (λ-TOA$_\mathrm{t}$) if

$$\lim_{z\to 1^-} L_\lambda(F(z)) < 1, \tag{2}$$

and *thermodynamically recurrent on the average with respect to L_λ* (λ-ROA$_\mathrm{t}$) if the limit is equal to 1.

The definition we counterpose is the following (and in both notations λ will often be tacitly understood).

Definition 2.6. Let (X, P) a random walk, and $\lambda=\{\lambda_n\}_{n\in\mathbb{N}}$ a sequence of probability measures on X. The random walk is *transient on the average with respect to L_λ* (λ-TOA) if $\underline{L}_\lambda(F)<1$, *recurrent on the average with respect to L_λ* (λ-ROA) if $\underline{L}_\lambda(F)=1$.

3. THE CLASSIFICATION ON THE AVERAGE

From now on, if not otherwise stated, we assume that $(X, E(X))$ is a connected (infinite), locally finite, nonoriented graph, that o is a fixed vertex of X, that (X, P) is a random walk, a priori not necessarily adapted to the graph $(X, E(X))$. Moreover, given $S \subseteq X$ we define $\partial S := \{x \in S : \exists y \notin S, (x, y) \in E(X)\}$.

3.1. Thermodynamical Classification and Classification on the Average

Some natural questions are: given a sequence λ is every random walk either λ-TOA$_t$ or λ-ROA$_t$? (Obviously, every random walk is λ-TOA or λ-ROA). Can we reverse the order of the two limits in Eq. (2)?

When do two sequences of probability measures give the same classification? In particular, if we consider the *averaging over balls with center o* (that is $\lambda_n(x) = \mathbb{1}_{B(o,n)}(x)/|B(o,n)|$), does the classification depend on the choice of $o \in X$?

Regarding the first question, it is not difficult to find examples of random walks with no thermodynamical classification.

Example 3.1. Let us consider the bihomogeneous tree $\mathbb{T}_{n,m}$ (which is the tree where the degrees of vertices are alternatively n and m). Despite its property of symmetry, the simple random walk on $\mathbb{T}_{n,m}$ (with $n \neq m$) is neither ROA$_t$ nor TOA$_t$ with respect to L_o, for any $o \in \mathbb{T}_{n,m}$ (but it is TOA with respect to any L_λ, see Example 3.14).

As for the second question, that is whether the limit in Eq. (2) coincides with $L_\lambda(F)$, in general the answer is no. Anyway, if the series $F(x,x)$ is totally convergent, (that is $\sum_{n \geqslant 1} k_n$ converges, where $k_n = \sup_{x \in X} f^{(n)}(x,x)$), then existence of the limit in (2) implies existence of $L_\lambda(F)$ and these limits coincide (Theorem 3.2(i) and (ii)). In other words, under this condition if the random walk is λ-TOA$_t$ (or λ-ROA$_t$) then it is also λ-TOA (or λ-ROA). Theorem 3.2 compares the two classifications on the average and provides some criteria.

Theorem 3.2. Let (X, P) be a random walk, ∞ be the point added to X in order to construct its one point compactification and λ a sequence of probability measure on X.

(i) If (X, P) is λ-ROA$_t$ then $L_\lambda(F)$ exists and is equal to 1, and the random walk is λ-ROA.

(ii) If $F(x,x)$ is a totally convergent series and (X, P) is λ-TOA$_t$ then $L_\lambda(F)$ exists, is strictly smaller than 1 and the random walk is λ-TOA.

(iii) (X, P) is λ-ROA $\Leftrightarrow$ for every $\varepsilon > 0$, $L_\lambda(\{x \in X : F(x,x) \geqslant 1 - \varepsilon\}) = 1$.

(iv) (X, P) is λ-TOA $\Leftrightarrow$ there exists $A \subseteq X$ such that $\bar{L}_\lambda(A) > 0$ and $\sup_A F(x,x) < 1$.

If λ is regular, the following hold:

(v) (X, P) is λ-ROA $\Leftrightarrow$ there exists $A \subseteq X$, such that $L_\lambda(A) = 1$ and $\lim_{\substack{x \to \infty \\ x \in A}} F(x, x) = 1$.

(vi) If there exists $A \subseteq X$ such that $\bar{L}_\lambda(A) > 0$ and $\lim_{\substack{x \to \infty \\ x \in A}} F(x, x) = \alpha < 1$ then the random walk is λ-TOA.

Proof. (i) For each fixed $x \in X$, the map $z \mapsto F(x, x \mid z)$ is a non-decreasing function bounded from above by 1, hence for some $\varepsilon \in (0, 1)$

$$1 = \lim_{z \to 1^-} \underline{L}_\lambda(F(z)) = \sup_{z \in (\varepsilon, 1)} \underline{L}_\lambda(F(z)) \leqslant \underline{L}_\lambda(F) \leqslant 1,$$

whence $L_\lambda(F) = 1$.

(ii) Since $F(x, x)$ is totally convergent, $g_z(k) = \sum_{x \in X} F(x, x \mid z)\, \lambda_k(x)$ converges, uniformly with respect to k, as z tends to 1, $|z| < 1$, to $g(k) = \sum_{x \in X} F(x, x)\, \lambda_k(x)$ (use Bounded Convergence Theorem). Thus by the Double Limit Theorem (Theorem 7.11 in ref. 16)

$$\lim_{\substack{z \to 1 \\ |z| < 1}} \lim_{k \to \infty} g_z(k) = \lim_{k \to \infty} g(k),$$

that is

$$\lim_{\substack{z \to 1 \\ |z| < 1}} L_\lambda(F(z)) = L_\lambda(F).$$

(iii) Let $A_\varepsilon^+ = \{x \in X : F(x, x) \geqslant 1 - \varepsilon\}$, $A_\varepsilon^- = (A_\varepsilon^+)^c$. Suppose that (X, P) is λ-ROA, then from

$$\sum_{x \in X} F(x, x)\, \lambda_n(x) \leqslant \lambda_n(A_\varepsilon^+) + (1 - \varepsilon)\, \lambda_n(A_\varepsilon^-),$$

taking the upper limit as n goes to infinity we get

$$0 \leqslant \bar{L}_\lambda(A_\varepsilon^-) \leqslant \varepsilon^{-1}(1 - \underline{L}_\lambda(F)) = 0,$$

whence $L_\lambda(A_\varepsilon^+) = 1$.

Now suppose that $L_\lambda(A_\varepsilon^+) = 1$ for all $\varepsilon \in (0, 1)$, then

$$\underline{L}_\lambda(F) \geqslant \underline{L}_\lambda(\mathbb{1}_{A_\varepsilon^+} F) \geqslant (1 - \varepsilon)\, \underline{L}_\lambda(A_\varepsilon^+) = 1 - \varepsilon,$$

whence $L_\lambda(F) = 1$ and (X, P) is λ-ROA.

(iv) It is a consequence of (iii) and of the fact that a random walk is either λ-ROA or λ-TOA.

(v) Suppose there exists such A and observe that $L_\lambda(F) = L_\lambda(\mathbb{1}_A F)$ (provided that the limit exists). The assertion follows from

$$\liminf_{\substack{x \to \infty \\ x \in A}} F(x, x) \leqslant \underline{L}_\lambda(F) \leqslant \bar{L}_\lambda(F) \leqslant \limsup_{\substack{x \to \infty \\ x \in A}} F(x, x). \tag{3}$$

Let us prove the first inequality in (3): by definition

$$q := \liminf_{\substack{x \to \infty \\ x \in A}} F(x, x) = \sup_{S \subseteq A \colon |S| < +\infty} \inf_{x \notin S} F(x, x),$$

thus for every $\varepsilon > 0$ there exists a finite S such that for every $x \notin S$, $F(x, x) > q - \varepsilon$ and hence, by regularity,

$$\sum_{x \in A} F(x, x)\, \lambda_n(x) \geqslant \sum_{x \in S} F(x, x)\, \lambda_n(x) + (1 - \lambda_n(S))(q - \varepsilon) \xrightarrow{n \to +\infty} q - \varepsilon,$$

whence $q \leqslant \underline{L}_\lambda(F)$. The other inequality is proven analogously.

Suppose now that (X, P) is λ-ROA and let $\{B_n\}_{n \in \mathbb{N}}$ be an increasing sequence of finite subsets of X such that $\bigcup_{n \in \mathbb{N}} B_n = X$. For any $n \in \mathbb{N}$, consider $A^+_{1/n}$. Let us construct recursively two increasing sequences $\{m_i\}_{i \in \mathbb{N}}$ and $\{n_i\}_{i \in \mathbb{N}}$ with values in $\mathbb{N}$ satisfying

$$\begin{cases} \lambda_m(A^+_{1/i}) > 1 - 1/i, & \forall m \geqslant m_i; \\ \lambda_m(A^+_{1/i} \cap B_{n_i}) > 1 - 1/i, & \forall m \colon m_i \leqslant m < m_{i+1}. \end{cases}$$

This is possible since $\lim_{m \to +\infty} \lambda_m(A^+_{1/i}) = 1$ for any $i \in \mathbb{N}$ and since (using Monotone Convergence Theorem) $\lim_{n \to +\infty} \lambda_m(A^+_{1/i} \cap B_n) = \lambda_m(A^+_{1/i}) > 1 - 1/i$ and the set $\{m \colon m_i \leqslant m < m_{i+1}\}$ is finite. We prove now that $A := \bigcup_{i=1}^{\infty} (A^+_{1/i} \cap B_{n_i})$ satisfies the two conditions in (iv).

If m satisfies $m_i \leqslant m < m_{i+1}$ then

$$\lambda_m(A) \geqslant \lambda_m(A^+_{1/i} \cap B_{n_i}) > 1 - 1/i$$

whence $\lim_{m \to +\infty} \lambda_m(A) = L_\lambda(A) = 1$.

By regularity we have that $L_\lambda(A \setminus B_{n_i}) = 1$ which implies that $A \setminus B_{n_i} \neq \emptyset$ and ∞ is an accumulation point for A. Moreover if $x \in A \setminus B_{n_i}$ we have that $x \in A^+_{1/j}$ for some $j > i$ and hence $F(x, x) > 1 - 1/j > 1 - 1/i$; which proves that $\lim_{\substack{x \to +\infty \\ x \in A}} F(x, x) = 1$.

(vi) It is a consequence of (v). ∎

Let us note that $F(x, x)$ needs not to be totally convergent even in the case of simple random walks (see Examples 3.4 and 3.12). Nevertheless, under certain conditions the series $F(x, x)$ is totally convergent.

Proposition 3.3. Let (X, P) be a random walk adapted to the graph $(X, E(X))$. If one of the following conditions holds then the series $F(x, x)$ is totally convergent.

(i) There exists a subgroup Γ of $\mathrm{AUT}(X)$ (the automorphism group of the graph) which acts quasi-transitively on X (i.e., with finitely many orbits), such that P is Γ-invariant.

(ii) The radius of convergence of the Green function $G(x, x \mid z)$ (which is independent of x) is $r > 1$ (in this case the random walk is λ-TOA and λ-TOA$_t$ for any sequence λ).

(iii) (X, P) is reversible (with reversibility measure m and total conductance $a(x, y) := m(x)\, p(x, y)$) and it satisfies the *strong isoperimetric inequality* that is

$$\sup_{A \subseteq X} \frac{m(A)}{s(A)} < +\infty,$$

where the supremum is taken over finite subsets A and $s(A) := \sum_{x \in A,\, y \in A^c} a(x, y)$.

Proof. (i) Let us pick a unique representative for each class of X/Γ and call the set of these vertices X_0. By hypotheses $k_n := \sup_{x \in X} f^{(n)}(x, x) = \max_{x \in X_0} f^{(n)}(x, x) \leqslant \sum_{x \in X_0} f^{(n)}(x, x)$. Hence $\sum_{n=0}^{\infty} k_n \leqslant \sum_{x \in X_0} F(x, x) \leqslant |X_0|$.

(ii) It follows from $f^{(n)}(x, x) \leqslant p^{(n)}(x, x) \leqslant 1/r^n$ which holds for every $x \in X$ and every $n \in \mathbb{N}$ (see Kingman[12]).

(iii) By Theorem 10.3 of ref. 18 we have that the strong isoperimetric inequality is equivalent to $r > 1$; then apply (ii) to conclude. ▮

For instance (i) applies to random walks adapted to Cayley graphs or of the simple random walk on quasi transitive graphs.

As for condition (ii), an example is given by a locally finite tree with minimum degree 2 and with finite upper bound on the lengths of its unbranched geodesics (see Theorems 10.9 in ref. 18).

Total convergence may be typically deduced from suitable upper estimates $p^{(n)}(x, x) \leqslant \varphi(n)$ for any $n \in \mathbb{N}$ and any $x \in X$. Such estimates can be obtained under geometrical and probabilistic conditions: Barlow *et al.*[1] (under some conditions for the volume growth), Grigor'yan and Telcs[9] (using conditions on the volume growth and the Green function). Other estimates, which depends on the volume of the ball, (see Coulhon and Grigory'an,[7] Grigor'yan and Telcs[8]) are useful to our purpose provided that we are able to give a (uniform) lower bound for the volume itself.

We observe that even if (X, P) is both thermodynamically classifiable and classifiable on the average, the two classifications may not agree, as is shown by the following example.

Example 3.4. Let $X := \bigcup_{n \in \mathbb{N}} \{n\} \times \mathbb{Z}_{n+1}$. For any $n, m \in \mathbb{N}$, $p \in \mathbb{Z}_{n+1}$, $q \in \mathbb{Z}_{m+1}$, (n, p) and (m, q) are neighbours if and only if one of the following holds (see Fig. 1)

(1) $p = 0_{\mathbb{Z}_{n+1}}$ and $q = 0_{\mathbb{Z}_{m+1}}$ and $|m-n| = 1$,

(2) $m = n$ and $p - q = \pm 1$, (where $p - q$ is the usual operation in $\mathbb{Z}_{n+1}$).

Given $\{p_n\}_{n \in \mathbb{N}}$ a $(0, 1)$-valued sequence such that $(p_n)^n \uparrow 1$ and $\alpha \in \mathbb{R}$, $0 < \alpha < 1/3$, we define the (adapted) transition probabilities as follows:

$$\begin{cases} p((0,0),(1,0)) = p((1,1),(1,0)) = 1, & \\ p((1,0),(1,1)) = p_1 + (1-p_1)\,\alpha, & \\ p((n,0),(n-1,0)) = (1-p_n)\,\alpha, & n \geqslant 1, \\ p((n,0),(n+1,0)) = (1-p_n)(1-2\alpha), & n \geqslant 1, \\ p((n,p),(n,p+1)) = p_n, & n \geqslant 2, \\ p((n,p),(n,p-1)) = (1-p_n), & n \geqslant 2, \quad p \neq 0, \\ p((n,0),(n,n-1)) = (1-p_n)\,\alpha, & n \geqslant 2. \end{cases}$$

By using standard stopping time arguments we easily see that this random walk is locally transient.

If we denote by $C_n := \{(n, p) : p \in \mathbb{Z}_{n+1}\}$, for any $x \in C_n$, we have that $f^{(n)}(x, x) \geqslant (p_n)^n$ and $f^{(m)}(x, x) \leqslant 1 - f^{(n)}(x, x)$ for all $m \neq n$. Thus $\lim_{x \to \infty} f^{(m)}(x, x) = 0$ for any $m \in \mathbb{N}$ and if $z \in (0, 1)$ by Bounded Convergence Theorem (using $z^m \geqslant f^{(m)}(x, x)\, z^m$) we derive $\lim_{x \to \infty} F(x, x \mid z) = 0$. Whence for any regular λ we obtain $L_\lambda(F(z)) = 0$ which implies that the random walk is λ-TOA$_t$.

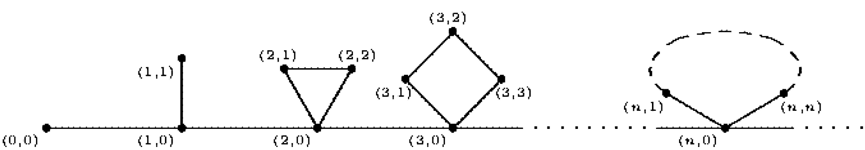

Fig. 1. A graph which is TOA$_t$ and ROA.

On the other hand $F(x,x) \geqslant f^{(m)}(x,x)$ for any $x \in X$, $m \in \mathbb{N}$, hence if $x \in \bigcup_{m \geqslant n} C_m$ we have that $F(x,x) \geqslant \inf_{m \geqslant n} (p_m)^m = (p_n)^n$ which implies $\lim_{x \to \infty} F(x,x) = 1$ and (always for any regular λ) $L_\lambda(F) = 1$ (that is, the random walk is λ-ROA). Since the classification on the average and the thermodynamical one are different this provides also an example of a random walk for which the series $F(x,x)$ is not totally convergent (Theorem 3.2(ii)).

3.2. Comparison of Different Averages

When do two different sequences $\{\lambda_n\}_{n \in \mathbb{N}}$ and $\{\eta_n\}_{n \in \mathbb{N}}$ induce the same classification? A particular case is the question of the independence of the L_o-classification on the average on the reference vertex o.

Proposition 3.5. Let $\lambda = \{\lambda_n\}_{n \in \mathbb{N}}$, $\eta = \{\eta_n\}_{n \in \mathbb{N}}$ two sequences of probability measures on X. Let us consider the following assertions:

(i) there exist two divergent sequences $\{i_n\}_{n \in \mathbb{N}}$ and $\{j_n\}_{n \in \mathbb{N}}$ of natural numbers, and two positive constants C, K such that $C\lambda_{i_n}(x) \geqslant \eta_n(x)$ and $K\eta_{j_n}(x) \geqslant \lambda_n(x)$ for every n and x;

(ii) for every $A \subseteq X$, $L_\lambda(A) = 1$ if and only if $L_\eta(A) = 1$;

(iii) a random walk (X, P) is ROA (respectively TOA) with respect to L_λ if and only if it is ROA (respectively TOA) with respect to L_η.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof. (i) $\Rightarrow$ (ii) It is easy (consider $L_\lambda(A^c)$ and $L_\eta(A^c)$).

(ii) $\Rightarrow$ (iii) By Theorem 3.2(iii) λ-ROA is equivalent to $L_\lambda(A_\varepsilon) = 1$ for all $\varepsilon > 0$, where $A_\varepsilon = \{x \in X : F(x,x) \geqslant 1 - \varepsilon\}$ and by hypothesis this is equivalent to $L_\eta(A_\varepsilon) = 1$. ∎

Consider now the case $\lambda_n(x) = \mathbb{1}_{B(o,n)}(x)/|B(o,n)|$. It has been shown in ref. 4 Section 4 that, if the graph has bounded geometry and

$$\lim_{n \to +\infty} \frac{|\partial B(o,n)|}{|B(o,n)|} = 0 \tag{4}$$

for some o, then the thermodynamical limit on the average is independent of the choice of o. This condition is not satisfied, for instance, by any homogeneous tree of degree greater than 2, or by any "fast growing" graph.

Applying Proposition 3.5 we find another topological condition (which is weaker than (4)) implying that the classification on the average (as defined in Definition 2.6) does not depend on the fixed vertex o.

Theorem 3.6. Let $(X, E(X))$ be such that there exists $x \in X$ satisfying

$$\sup_{n \in \mathbb{N}} \frac{|S(x, n+1)|}{|B(x, n)|} < +\infty, \tag{5}$$

(where $S(x, n+1)$ is the sphere centered in x with radius $n+1$) then the L_o-classification on the average of any random walk is independent of the choice of o.

Proof. Let o and o' be two reference vertices in X, $d(o, o') = l$, and put $\lambda_n(x) = \mathbb{1}_{B(o,n)}(x)/|B(o, n)|$, $\eta_n(x) = \mathbb{1}_{B(o',n)}(x)/|B(o', n)|$. From (5) one deduces that, for n sufficiently large, $\eta_n(x) \leqslant C\lambda_{l+n}(x)$. The assertion follows from Proposition 3.5. ▮

We observe that (5) is satisfied by any graph with bounded geometry. On the other hand, bounded geometry is not necessary, as is shown by the following example.

Example 3.7. Given a vertex x_0, construct the tree T as follows: x_0 has one neighbour in $S(x_0, 1)$ and each vertex in $S(x_0, m)$ has exactly one neighbour in $S(x_0, m+1)$ if $m \neq k^2$ for all $k = 1, 2, \ldots$, and exactly k neighbours if $m = k^2$. Then T satisfies Eq. (5) and does not have bounded geometry.

3.3. Local Classification and Classification on the Average

Let us now make some comparisons between the local classification and the classification on the average of a random walk. The following remark is obvious.

Remark 3.8. If (X, P) is locally recurrent, then $L_\lambda(F) = 1$ and the random walk is λ-ROA for any given sequence λ.

Example 3.4, which is locally transient, λ-TOA$_t$ and λ-ROA, shows that local transience does not imply transience on the average. The following example shows that local transience does not imply thermodynamical transience on the average.

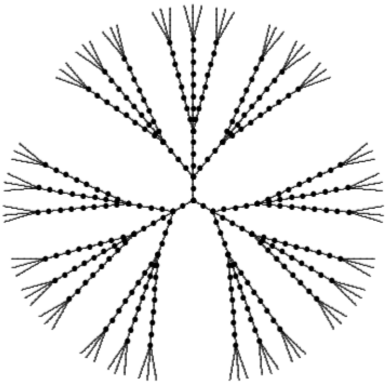

Fig. 2. A slow-growing tree.

Example 3.9. Given the sequence of natural numbers $\{s_j = \sum_{i=1}^{j} \beta^i\}_{j \geqslant 1}$, where $\beta \geqslant 2$ is an integer number, $s_0 = 0$ and o is the root, the construction of the tree T is similar to the one in Example 3.7. Each element on the sphere $S(o, m)$ has exactly one neighbour on the sphere $S(o, m+1)$ if $m \neq s_j$ for any $j \geqslant 0$ and exactly α neighbours if $m = s_j$ $(\alpha \in \mathbb{N})$ (Fig. 2 represents the case $\alpha = 3$, $\beta = 2$). An application of Theorem 2.1 proves that T is locally transient if and only if $\alpha > \beta$ (see, for instance, Remark 4.3 in ref. 20).

Consider $\lambda_n(x) = \mathbb{1}_{B(o,n)}(x)/|B(o,n)|$; it is easy to prove that the set A obtained by removing from X the balls of radius k centered in the elements of $S(o, s_k)$, for all $k \in \mathbb{N}$, has L_o-measure equal to 1. Moreover on A, for every fixed n, as x tends to infinity $f^{(n)}(x, x)$ is definitively equal to $f_{\mathbb{Z}}^{(n)}$ (the first time return probabilities of the simple random walk on $\mathbb{Z}$, which do not depend on the starting vertex). Hence as in (3) one can show that $L_o(f^{(n)}(\cdot,\cdot)) = L_o(f_{\mathbb{Z}}^{(n)}(\cdot,\cdot))$. Then using Fubini and Bounded Convergence Theorem,

$$\sum_{x \in X} \sum_{j} f^{(j)}(x, x)\, z^j \lambda_n(x)$$
$$= \sum_{j} \sum_{x \in X} f^{(j)}(x, x)\, z^j \lambda_n(x) \xrightarrow{n \to \infty} L_\lambda(F_{\mathbb{Z}}(z)) \xrightarrow{z \to 1^-} 1.$$

Thus the graph is $\mathrm{ROA}_{\mathrm{t}}$ (and ROA) with respect to L_o with any reference vertex o.

It is known that (local) transience is equivalently expressed by a condition on the Green function: $G(x, x) := G(x, x \mid 1) = +\infty$ for some (i.e., for every) $x \in X$. In the average case we can only claim a partial result.

Proposition 3.10. Let (X, P) be a random walk. Then:

(i) if the random walk is λ-ROA$_{\mathrm{t}}$ then $\lim_{z\to 1^-} \underline{L}_\lambda(G(z)) = +\infty$;

(ii) if the random walk is λ-ROA then $L_\lambda(G) = +\infty$.

Proof. Let $\varphi(t) := 1/(1-t)$, it is known (see ref. 18) that $G(x, x \mid z) = \varphi(F(x, x \mid z))$ for all $x \in X$ and for all $z \in \mathbb{R}$, $|z| < 1$. By Jensen's inequality

$$\varphi\left(\sum_{x\in X} F(x, x \mid z)\, \lambda_n(x)\right) \leqslant \sum_{x\in X} G(x, x \mid z)\, \lambda_n(x).$$

If we take the limit as n goes to infinity of both sides of the previous equation, taking into account the continuity of φ,

$$\begin{aligned}\varphi(L_\lambda(F(z))) &= \lim_{n\to+\infty} \varphi\left(\sum_{x\in X} F(x, x \mid z)\, \lambda_n(x)\right)\\ &\leqslant \liminf_{n\to+\infty} \sum_{x\in X} G(x, x \mid z)\, \lambda_n(x) =: \underline{L}_\lambda(G(z));\end{aligned}$$

hence

$$\lim_{z\to 1^-} \underline{L}_\lambda(G(z)) \geqslant \liminf_{z\to 1^-} \varphi(L_\lambda(F(z))) = +\infty.$$

The case $L_\lambda(F) = 1$ is completely analogous (note that $\sum_{x\in X} G(x, x)\, \lambda_n(x)$ may be equal to $+\infty$ for some $n \in \mathbb{N}$). ∎

Observe that in Proposition 3.10(i) existence of $L_\lambda(G(z))$ is not guaranteed and then we have to consider $\underline{L}_\lambda$ instead. Also notice that reversed implications are not true, see for instance Example 3.11, which in ref. 4 is called a *mixed* λ-TOA$_{\mathrm{t}}$ graph. It provides an example of random walk which is λ-TOA even if $L_\lambda(G) = +\infty$. Moreover Example 3.11 is TOA if we average over balls, but it is ROA if we average over a suitable family of subsets.

Example 3.11. Let X be the graph obtained from $\mathbb{Z}^3$ by deleting all horizontal edges joining vertices with positive height (compare with ref. 4 where this graph is an example of mixed λ-TOA$_{\mathrm{t}}$ and see Fig. 3): we call X_+ the set of vertices with (strictly) positive height and $X_- = X_+^c$. The simple random walk on X is locally transient (Theorem 2.1), indeed there is a finite energy flow u defined on $E(\mathbb{Z}^3)$ from the origin o to ∞ with input 1. By Corollary 4.7 we have that the simple random walk is TOA if we average over balls and $L_o(G) = +\infty$. More generally it is TOA with respect to any L_λ such that $\bar{L}_\lambda(X_-) > 0$ and $L_\lambda(G) = +\infty$ if and only if $L_\lambda(X_+) > 0$.

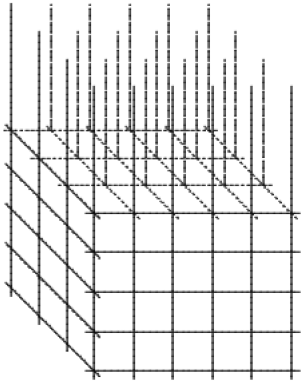

Fig. 3. A graph whose classification depends on λ.

The same graph is ROA$_t$ (thus ROA) with respect to L_λ with $\lambda_n = \mathbb{1}_{B_n}(x)/|B_n|$, where $\{B_n = (B(o, 2^n) \cap X_+) \cup (B(o, n) \cap X_-)\}_{n \in \mathbb{N}}$. Indeed $L_\lambda(X_-) = 0$ and, if we denote by $X_n = \{x \in X : d(x, X_-) \leqslant n\}$, then $L_\lambda(X_n) = 0$ for all $n \in \mathbb{N}$. Moreover $f^{(j)}(x, x) = f^{(j)}_{\mathbb{Z}}$ for all $j \in \mathbb{N}$, $x \in X_j^c$ (and $f^{(j)}(x, x) \leqslant f^{(j)}_{\mathbb{Z}}$ if $x \in X_j$). Then using Fubini and Bounded Convergence Theorem,

$$\sum_{x \in X} \sum_j f^{(j)}(x, x)\, z^j \lambda_n(x)$$

$$= \sum_j \sum_{x \in X_j} f^{(j)}(x, x)\, z^j \lambda_n(x)$$

$$+ \sum_j \sum_{x \in X_j^c} f^{(j)}(x, x)\, z^j\, \lambda_n(x) \xrightarrow{n \to \infty} L_\lambda(F_{\mathbb{Z}}(z)) \xrightarrow{z \to 1^-} 1.$$

The following example was suggested by D. Cassi, R. Burioni, and A. Vezzani as an example of random walk which is locally recurrent (thus ROA with respect to any L_λ) but λ-TOA$_t$ for any suitable choice of λ.

Example 3.12. Let X be the graph obtained by attaching at each vertex i of $\mathbb{N}$ a cube lattice C_i of side n_i by one of its corners (in Fig. 4, $n_i = i$). Suppose that n_i diverges. Then the simple random walk on X is locally recurrent (according to Theorem 2.1), hence it is also ROA with respect to any L_λ.

Let $B_n = \bigcup_{i=1}^n C_i$ and consider $\lambda_n(x) = \mathbb{1}_{B_n}(x)/|B_n|$. The simple random walk on X is λ-TOA$_t$. Indeed for each $k \in \mathbb{N}$, let X_k be the set of all the vertices at distance at most k from the surface of the cubes. Then

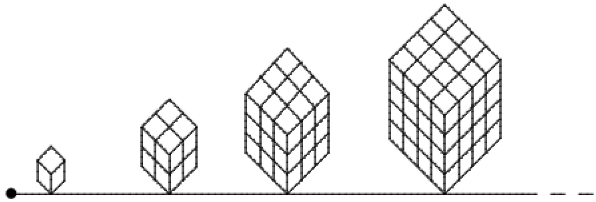

Fig. 4. A locally recurrent but TOA_t graph.

$L_\lambda(X_k) = 0$ and $f^{(k)}(x, x) = f^{(k)}_{\mathbb{Z}^3}$ for all $x \in X^c_k$, where $f^{(k)}_{\mathbb{Z}^3}$ is the k-step first time returning probability in $\mathbb{Z}^3$. The rest of the proof is analogous to the one for the previous example.

We recalled in Section 1 that Theorem 2.1 gives a useful tool to (locally) classify reversible random walks. A similar result can be stated for the classification on the average.

Theorem 3.13. Let (X, P) be a reversible random walk, with reversibility measure m satisfying $\inf m(x) > 0$, $\sup m(x) < +\infty$ (in particular this condition is satisfied by the simple random walk on a graph with bounded geometry). Then, for any given sequence λ, the following are equivalent:

(a) the random walk is λ-TOA;

(b) there exists $A \subseteq X$ such that: $\bar{L}_\lambda(A) > 0$, for all $x \in X$, $i_0 \neq 0$ there is a finite energy flow u^x from x to ∞ with input i_0 and $\sup_{x \in A} \langle u^x, u^x \rangle < +\infty$.

Proof. We consider (X, P) (locally) transient (otherwise the random walk is λ-ROA).

(a) $\Rightarrow$ (b) Recall that $u^x = -\frac{i_0}{m(x)} \nabla G(\cdot, x)$ is a finite energy flow from x to ∞ with input i_0 and energy

$$\langle u^x, u^x \rangle = \frac{i_0^2}{m(x)} G(x, x), \tag{6}$$

(where ∇ denotes the difference operator, see Theorem 2.12 in ref. 18). But by Theorem 3.2(iv) the network is λ-TOA if and only if there exists $\alpha < 1$, $A \subseteq X$ such that $\bar{L}_\lambda(A) > 0$ and $F(x, x) \leqslant \alpha$ for every $x \in A$. Since $G(x, x) = 1/(1 - F(x, x))$ this is equivalent to $\sup_{x \in A} G(x, x) < +\infty$. By Eq. (6) and our hypotheses on the reversibility measure, this implies (b).

(b) ⇒ (a) It suffices to show that $\sup_{x \in A} G(x, x) < +\infty$, but this is a consequence of the estimate

$$G(x, x) \leqslant \frac{m(x)\langle u^x, u^x \rangle}{i_0^2},$$

which can easily be deduced from the proof of Theorem 2.12 in ref. 18. ∎

As an application we classify bihomogeneous trees and a whole family of inhomogeneous trees.

Example 3.14. Consider the bihomogeneous tree $\mathbb{T}_{m,n}$ and a couple of vertices x_n and x_m, the first with degree n and the second with degree m (in Fig. 5, $m = 3$ and $n = 2$). There are two finite energy flows u^n and u^m with fixed input i_0, respectively from x_n to infinity and from x_m to infinity. Thus, translating u^n or u^m (depending on the degree of x), we obtain a finite energy flow from any vertex x to infinity (with input i_0). By Theorem 3.13 this proves that the simple random walk on $\mathbb{T}_{m,n}$ is λ-TOA (the for any λ). Moreover, since $L_o(F)$ does not exist for any reference vertex o, and the series $F(x, x)$ is totally convergent by Proposition 3.3(i) (with $\Gamma = AUT(X)$), then by Theorem 3.2(i) and (ii) the simple random walk on $\mathbb{T}_{m,n}$ is not thermodynamically classifiable.

Analogously one shows that the simple random walk on a tree $T'_{k,n}$ whose vertices have degree 2 or k ($k \geqslant 3$) and such that the distance between ramifications is n ($n \geqslant 2$) is λ-TOA (while in the preceding case we had only two flows, here we have at most $[n/2]+1$ flows).

Now consider an inhomogeneous tree $T''_{k,n}$ whose vertices have degree 2 or k ($k \geqslant 3$) and such that the distance between ramifications does not exceed n ($n \geqslant 2$): see Fig. 6 for the case $k = 3$ and $n = 2$. Here the family of

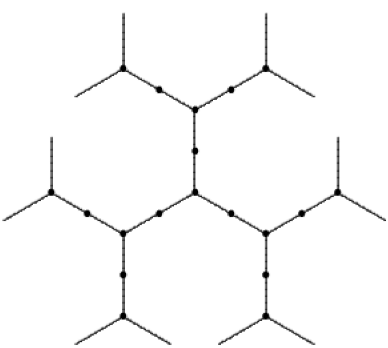

Fig. 5. A bihomogeneous tree.

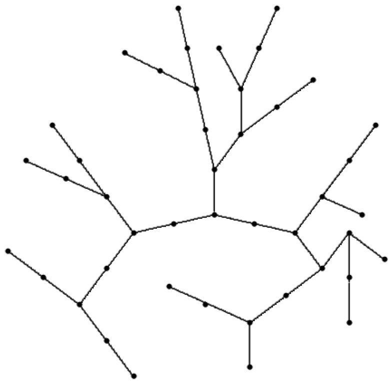

Fig. 6. A nonhomogeneous tree.

finite energy flows (depending on the vertex chosen for the input) can be easily constructed from the family of flows of $T'_{k,n}$ (by deleting an appropriate subset of vertices) in such a way that the supremum of the energies on $T''_{k,n}$ is bounded from above by the maximum of the energies on $T'_{k,n}$.

4. SUBGRAPHS AND GRAPHS

In this section we study information which can be inferred from the knowledge of the behaviour of random walks on subgraphs.

Since we want to average on a subgraph, the first thing to do is to normalize the weights.

Definition 4.1. Let $\lambda = \{\lambda_n\}_{n \in \mathbb{N}}$ be a sequence of probability measures on X and $S \subseteq X$ such that $\lambda_n(S) > 0$, for all $n \in \mathbb{N}$. Then the limit on the average L_λ^S defined on S by $\lambda_n^S := \lambda_n|_S / \lambda_n(S)$ for every $n \in \mathbb{N}$ and $x \in S$ is called normalized limit on the average (analogously for $\bar{L}_\lambda^S$ and $\underline{L}_\lambda^S$).

The following straightforward lemma links L_λ^S and L_λ.

Lemma 4.2. Let $S \subseteq X$ be an L_λ-measurable subset with positive L_λ-measure and f a function defined on X, then:

(i) $f|_S \in \mathscr{D}(L_\lambda^S) \Leftrightarrow \mathbb{1}_S f \in \mathscr{D}(L_\lambda)$,

(ii) if $f|_S \in \mathscr{D}(L_\lambda^S)$ then $L_\lambda(\mathbb{1}_S f) = L_\lambda^S(f|_S) \cdot L_\lambda(S)$,

(iii) $\underline{L}_\lambda(\mathbb{1}_S f) = \underline{L}_\lambda^S(f|_S) \cdot L_\lambda(S)$ and $\bar{L}_\lambda(\mathbb{1}_S f) = \bar{L}_\lambda^S(f|_S) \cdot L_\lambda(S)$.

Let us consider the average of the generating function F related to a random walk (X, P). There are two different ways of looking at the behaviour of the random walk on a subgraph $S \subseteq X$. The first one is to consider S as a subset of the graph with $F(x, x)$ restricted to the sites in S. The second approach is to view S as an independent graph, with possibly different generating functions $F(x, x)$.

We start with the first point of view. Lemma 4.2 implies that sets of measure zero have no weight in the averaging procedure (think of S such that $L_\lambda(S) = 1$: then $L_\lambda^S(F|_S) = L_\lambda(F)$); hence for the L_o-classification slowlier growing subgraphs are not influent.

Remark 4.3. Let $X = X_1 \cup X_2$, where $X_1 \cap X_2$ is finite. Suppose that $|B(o, n) \cap X_1| / |B(o, n) \cap X_2| \to 0$ as n goes to infinity (X_1 grows slowlier than X_2). Then $L_o(X_1) = 0$ and the L_o-classification of any random walk on X depends only on the restriction of the generating function F on X_2.

For the local classification of an irreducible random walk the existence of a vertex x such that $F(x, x) < 1$ guarantees transience. The following corollary of Lemma 4.2 is an analog for the classification on the average.

Corollary 4.4. Let (X, P) be a random walk and let $S \subseteq X$ such that $L_\lambda(S) > 0$. If the restriction of F to S satisfies $\underline{L}_\lambda^S(F|_S) < 1$ then (X, P) is λ-TOA.

Proof. Let $\{n_j\}_{j \in \mathbb{N}}$ be such that

$$\lim_{j \to \infty} \sum_{x \in S} F(x, x)\, \lambda_{n_j}(x) = \underline{L}_\lambda^S(F|_S) < 1$$

and let $\eta_j := \lambda_{n_j}$. Then $L_\eta(S) = L_\lambda(S) > 0$, $L_\eta^S(F|_S) = \underline{L}_\lambda^S(F|_S)$ and

$$\begin{aligned}\underline{L}_\lambda(F) \leqslant \underline{L}_\eta(F) &= L_\eta^S(F|_S)\, L_\eta(S) + \underline{L}_\eta^{S^c}(F|_{S^c})\, L_\eta(S^c)\\ &\leqslant 1 - (1 - \underline{L}_\lambda^S(F|_S))\, L_\lambda(S) < 1. \quad \blacksquare\end{aligned}$$

Similarly, under certain conditions, one can obtain the precise value of $L_\lambda(F)$ from the limiting values of F on each subgraph of a partition of X.

Lemma 4.5. Let $\bar{X} := X \cup \{\infty\}$ be the one point compactification of X and let $\{A_i\}_{i=1}^k$ be a partition of X such that each A_i is L_λ-measurable. Suppose that for every i such that $L_\lambda(A_i) > 0$ there exists $\lim_{x \to \infty} F|_{A_i}(x, x) = \alpha_i$. Then, for any regular λ, $L_\lambda(F)$ exists and is equal to $\sum_{i=1}^k L_\lambda(A_i)\, \alpha_i$.

The proof is analogous to Theorem 3.2(vi) and uses Lemma 4.2. Moreover the same result can be proven with a countable partition, provided that $\sum_{i \in \mathbb{N}} F(x,x)\, \mathbb{1}_{A_i}(x)$ converges uniformly to $F(x,x)$ with respect to $x \in X$ (uniformity is exploited as in Theorem 3.2(ii)).

We remark that even though sets of measure zero have no influence on the resulting limit on the average of the function F their presence may change the return probabilities and hence the function F that we average. This is the main difficulty of the second approach.

We know that local transience (even if on the whole graph) does not imply that the random walk is λ-TOA, but under certain regularity conditions local transience of the simple random walk on a subgraph (regarded as an independent graph) implies that the simple random walk on the whole graph is λ-TOA.

Theorem 4.6. Let λ be a sequence of probability measures on X and $(A, E(A))$ be a subgraph of X such that $\bar{L}_\lambda(A) > 0$. Suppose that there exists $x_0 \in A$ such that for every vertex $y \in A$ there exists an injective map $\gamma_y \colon A \to A$ such that (i) $\gamma_y(x_0) = y$ and (ii) for any $w, z \in A$, $(w,z) \in E(A)$ implies $(\gamma_y(w), \gamma_y(z)) \in E(A)$. If the simple random walk on $(A, E(A))$ is transient then the simple random walk on $(X, E(X))$ is λ-TOA.

Proof. Let us consider $(A, E(A))$ with the edge orientation induced by X. Let u be a flow on $(A, E(A))$ with finite energy starting from x_0 to infinity with input 1. Given any simple random walk, the conductance is the characteristic function of the edges, hence for any $y \in A$ it is easy to show that the following equation

$$u_y(a,b) := \begin{cases} \varepsilon_{\gamma_y}(a,b)\, u(\gamma_y{}^{-1}(a), \gamma_y{}^{-1}(b)) & \text{if } (a,b) \in E(A); \\ 0 & \text{if } (a,b) \notin E(A); \end{cases} \qquad \forall (a,b) \in E(X)$$

(where $\varepsilon_{\gamma_y}(a,b)$ is equal to $+1$ or -1 according to $(\gamma_y(a), \gamma_y(b))^+ = (x,y)^+$ or not) define a flow u_y on $(X, E(X))$ starting from y to ∞ with input one and with the same energy as u. Apply now Theorem 3.13. ∎

We observe that the condition on A in the previous statement is a requirement of "self-similarity" of A (take for instance Cayley graphs).

Corollary 4.7. Let λ be a sequence of probability measures on X and $(G, E(G))$ be a Cayley graph and $A \subseteq G$ such that (i) the group identity $e \in A$, (ii) for any $x, y \in A$ we have that $xy \in A$, and (iii) the simple random walk on $(A, E(A))$ is transient. If $(X, E(X))$ is a locally finite graph which contains $(A, E(A))$ as a subgraph and $\bar{L}_\lambda(A) > 0$ then the simple random walk on $(X, E(X))$ is λ-TOA.

Now we want to partition X into two subgraphs A and B and deduce average properties of an adapted random walk on X from the behaviour of its restrictions to A and B. The first question is what is meant as restriction of a random walk to a subgraph.

Definition 4.8. Let $(S, E(S))$ be a subgraph on $(X, E(X))$ and let (X, P) be a random walk on $(X, E(X))$. A random walk (S, P_S) is called induced random walk on S if for every $x \in S \setminus \partial S$ and every $y \in S$ we have that $p_S(x, y) = p(x, y)$.

We note that in general the induced random walk is not uniquely determined (there are different choices on ∂S), but if $n \in \mathbb{N}$ and $x \in S$ are such that $d(x, \partial S) \geqslant n$ then

$$f^{(n)}(x, x) = f_S^{(n)}(x, x), \qquad p^{(n)}(x, x) = p_S^{(n)}(x, x). \tag{7}$$

We require that ∂A and ∂B have L_λ-measure zero. Then we need that $f^{(n)}$ and $f_A^{(n)}$ coincide outside a negligible set and this can be proven for the L_o-measure.

Lemma 4.9. Let $(X, E(X))$ be a graph with bounded geometry and C a subset of X such that $L_o(C) = 0$. Let $X_n = \{x \in X : d(x, C) \leqslant n\}$, then $L_o(X_n) = 0$ for every $n \in \mathbb{N}$.

Proof. We note that for every $n, r \in \mathbb{N}$, $X_n \cap B(o, r) \subseteq \bigcup_{x \in C \cap B(o, n+r)} B(x, n)$ and by hypotheses,

$$|B(o, r+n)|/|B(o, r)| \leqslant \sup_{x \in X} |B(x, n)| =: M.$$

Then

$$\frac{|X_n \cap B(o, r)|}{|B(o, r)|} \leqslant M \frac{|C \cap B(o, r+n)|}{|B(o, r)|} \leqslant M^2 \frac{|C \cap B(o, r+n)|}{|B(o, r+n)|} \xrightarrow{r \to +\infty} 0. \qquad \blacksquare$$

To prove our result for the thermodynamical classification we need a technical lemma which claims that two generating functions with coefficients that coincide but for a zero L_λ-measure set, have the same L_λ limit.

Lemma 4.10. Let $H_1(x, z) = \sum_{n=0}^{\infty} a_n(x) z^n$, $H_2(x, z) = \sum_{n=0}^{\infty} b_n(x) z^n$, where a_n and b_n are nonnegative functions on X, for all $n \in \mathbb{N}$. Suppose that $\sum_{n=0}^{\infty} k_n z^n$, where $k_n = \max\{\sup_{x \in X} a_n(x), \sup_{x \in X} b_n(x)\}$, has positive radius r of convergence. Moreover suppose that for every $n \in \mathbb{N}$ there exists

$X_n \subseteq X$ such that $L_\lambda(X_n)=0$ and $a_n(x)=b_n(x)$ for all $x \in X_n^c$. Then for all $z \in \mathbb{C}$, $|z|<r$, $H_1(\cdot, z) \in \mathscr{D}(L_\lambda)$ if and only if $H_2(\cdot, z) \in \mathscr{D}(L_\lambda)$ and

$$L_\lambda(H_1(z)) = L_\lambda(H_2(z)),$$

(the same assertion holds also with $\bar{L}_\lambda$ or $\underline{L}_\lambda$).

Proof. Let us choose z such that $|z|<r$. Using Bounded Convergence Theorem it is not difficult to prove that

$$\lim_{n\to\infty} \sum_{i=0}^{\infty} \sum_{x \in X_i} a_i(x)\, z^i \lambda_n(x) = 0.$$

Hence it is obvious that (if the limit exists)

$$L_\lambda(H_1(z)) = \lim_{n\to\infty} \sum_{i=0}^{\infty} \sum_{x \in X_i^c} a_i(x)\, z^i \lambda_n(x).$$

But by the hypotheses $a_i(x)=b_i(x)$ on X_i^c, whence the limit coincides with $L_\lambda(H_2(z))$. ∎

In the next theorem, for a general λ, one could require $L_\lambda(X_n)=0$ for all n, but for the sake of briefness we treat only the case of the L_o-classification. Moreover, in this particular case one has to pay attention to the role played by the metric d which appears in Eq. (7) as well as in the definition of the balls. To avoid further complications we assume that X is partitioned in two subgraphs A and B which are *starlike*. A subgraph A is *starlike* if there exists $o' \in A$ such that for every $x \in A$ at least one geodesic path (in X) from o' to x lies in A (thus we are sure that $d_A(o', x) = d_X(o', x)$ and we denote this distance simply by d). Under this hypothesis, $L_{o'}(f)$ considered with d_A is the same as $L_{o'}(f)$ considered with d_X. Moreover, if $(X, E(X))$ has bounded geometry, these limits coincide with $L_o(f)$ for all $o \in A$.

To simplify notations, we also require that $A \cap B = \{o\}$, but this is no severe restriction, since it implies that we choose $o \in \partial A$, $B' = B \cup \{o\}$ and we redefine $B = B'$.

Theorem 4.11. Let $(X, E(X))$ be an infinite graph with bounded geometry and let $(A, E(A))$, $(B, E(B))$ be two subgraphs such that $A \cup B = X$, $\{o\} = A \cap B$, and A, B are starlike. Moreover suppose that $L_o(A)>0$, $L_o(B)>0$, and $L_o(\partial A)=0$. Let P be a stochastic matrix representing a random walk on X (adapted to $(X, E(X))$) and let us consider two induced random walks (represented by P_A and P_B) on the subgraphs $(A, E(A))$ and $(B, E(B))$. The following hold:

(i) if any two of (X, P), (A, P_A), (B, P_B) is L_o-thermodynamically classifiable, then so is the third.

If (X, P), (A, P_A), (B, P_B) are L_o-thermodynamically classifiable, then

(ii) (A, P_A) is $\mathrm{TOA_t}$ implies that (X, P) is $\mathrm{TOA_t}$;

(iii) if $L_o(A) < 1$ then (X, P) is $\mathrm{ROA_t}$ if and only if (A, P_A) and (B, P_B) are both $\mathrm{ROA_t}$.

Proof. (i) We note that since $\deg(\cdot)$ is bounded, we have that $L_o(\partial A) = 0$ if and only if $L_o(\partial B) = 0$. Since

$$F(z) - \mathbb{1}_A F(z) - \mathbb{1}_B F(z) + \mathbb{1}_{\{o\}} F(z) = 0, \tag{8}$$

by hypothesis at least three of these functions belongs to $\mathscr{D}(L_o)$ and hence also the remaining one does. Lemma 4.10 yields to the conclusion.

(ii) Let F and F_A be the generating functions (depending on $x \in X$ and $z \in [0, 1)$) of the hitting probabilities associated to P and P_A respectively. By the hypotheses $F(z) \in \mathscr{D}(L_o)$ and $F_A(z) \in \mathscr{D}(L_o^A)$ for all $z \in (\varepsilon, 1)$, for some $\varepsilon \in (0, 1)$. By Eq. (7) and Lemma 4.9 we have that $f^{(n)}$ and $f_A^{(n)}$ coincide on A except for a L_o^A-negligible set, hence we can apply Lemma 4.10 to F_A and $F_{|A}$ obtaining that $F_{|A}(z) \in \mathscr{D}(L_o^A)$ and $L_o^A(F_{|A}(z)) = L_o^A(F_A(z))$ for all $z \in (\varepsilon, 1)$. By Lemma 4.2 we have that $L_o^A(F_A(z)) = L_o^A(F_{|A}) = L_o(\mathbb{1}_A F(z))/L_o(A)$. From Eq. (8) we get

$$L_o(F(z)) \leqslant L_o^A(F_A(z))\, L_o(A) + (1 - L_o(A)), \tag{9}$$

whence if $\lim_{z \to 1^-} L_o^A(F_A(z)) < 1$ it follows that $\lim_{z \to 1^-} L_o(F(z)) < 1$.

(iii) The *only if* part is a consequence of (ii). As for the *if* part, define $C = \partial A \cup \partial B$ and $X_n = \{x \in X : d(x, C) \leqslant n\}$. Then if $x \in A \setminus X_n$, $f^{(n)}(x, x) = f_A^{(n)}(x, x)$ and if $x \in B \setminus X_n$, $f^{(n)}(x, x) = f_B^{(n)}(x, x)$. Applying Lemmas 4.10 and 4.2

$$L_o(F(z)) = L_o^A(F_A(z))\, L_o(A) + L_o^B(F_B(z))\, L_o(B) \xrightarrow{z \to 1^-} L_o(A) + L_o(B) = 1. \quad \blacksquare$$

The previous theorem is different from those in ref. 4 since here a subgraph A is regarded as an independent graph with an induced random walk. In ref. 4, one is supposed to study the generating function F of (X, P) to classify the random walk; in our approach one can study independently two (hopefully) simpler random walks P_A and P_B (on A and B respectively) and then the classification of the main random walk can be inferred.

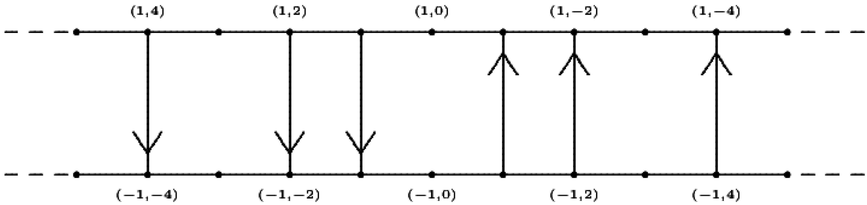

Fig. 7. Two "bridged" chains.

Unfortunately, in general a similar result does not hold for the classification on the average we introduced in this paper, as is shown by the following example.

Example 4.12. Let X be the graph in Fig. 7, constructed by two copies of $\mathbb{Z}$: the vertices of the upper copy are denoted by $(1, i)$ and the ones of the lower copy by $(-1, i)$ $(i \in \mathbb{Z})$. Vertices $(1, 2^i)$, $i = 0, 1, \ldots$, are linked by an (oriented) edge with vertices $(-1, -2^i)$ and so are vertices $(-1, 2^i)$ and $(1, -2^i)$. Given a fixed $\varepsilon \in (0, 1/2)$, the random walk is described as follows: in the upper copy of $\mathbb{Z}$ there is a probability $1-\varepsilon$ of moving leftwards and ε of moving rightwards, but for the vertices $(1, 2^i)$, where the walker may move to $(1, 2^i+1)$, $(1, 2^i-1)$, $(-1, -2^i)$ with probability $(1-\varepsilon)/2$, $\varepsilon/2$, and $1/2$ respectively. The transition probabilities on the lower copy of $\mathbb{Z}$ are then obtained by symmetry (hence with a rightward drift). By standard arguments one can prove that (X, P) is recurrent (hence λ-ROA for any λ). Besides, if we cut all the "bridges" (which are a set of L_o-measure zero) and redefine the transition probabilities on the two disjoint copies of $\mathbb{Z}$ such that $p(i, i+1) = 1-\varepsilon$ and $p(i, i-1) = \varepsilon$ $(i \in \mathbb{Z})$, it is clear that these two random walks are transient and TOA with respect to L_o.

5. CONCLUSIONS

In this paper we proposed a new classification of random walks and we showed how to manage it from a technical point of view.

It seems reasonable to consider also the "dual" classification obtained by substitution in Definition 2.2 of the $\underline{L}_\lambda$ with the $\bar{L}_\lambda$.

Definition 5.1. Let (X, P) be a random walk and $\lambda = \{\lambda_n\}_{n \in \mathbb{N}}$ a sequence of probability measures on X. The random walk is $\bar{\lambda}$-ROA (respectively $\bar{\lambda}$-TOA) if and only if $\bar{L}_\lambda(F) = 1$ (resp. $\bar{L}_\lambda(F) < 1$).

We observe that there are more random walks which are $\bar{\lambda}$-ROA than random walks which are λ-ROA. Moreover a random walk is $\bar{\lambda}$-TOA if and only if there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that for any $n \geqslant n_0$ we have $\sum_{x \in X} F(x, x)\, \lambda_n(x) < 1 - \varepsilon$.

With slight differences, it is possible to prove similar results for this new classification. For instance we state the analogues of Theorems 3.2 and 3.13.

Theorem 5.2. Let (X, P) be a random walk and let ∞ be the point added to X in order to construct its one point compactification. Let λ be a regular limit on the average.

(i) If (X, P) is λ-ROA$_t$ then $L_\lambda(F) = 1$ and the random walk is $\bar{\lambda}$-ROA.

(ii) If $F(x, x)$ is a totally convergent series and (X, P) is λ-TOA$_t$ then $L_\lambda(F) < 1$ and the random walk is $\bar{\lambda}$-TOA.

(iii) (X, P) is $\bar{\lambda}$-ROA $\Leftrightarrow$ for every $\varepsilon > 0$, $\bar{L}_\lambda(\{x \in X : F(x, x) \geqslant 1 - \varepsilon\}) = 1$.

(iv) (X, P) is $\bar{\lambda}$-ROA $\Leftrightarrow$ there exists $A \subseteq X$, such that $\bar{L}_\lambda(A) = 1$ and $\lim_{\substack{x \to \infty \\ x \in A}} F(x, x) = 1$.

(v) (X, P) is $\bar{\lambda}$-TOA $\Leftrightarrow$ there exists $A \subseteq X$ such that $\underline{L}_\lambda(A) > 0$ and $\sup_A F(x, x) < 1$.

(vi) If there exists $A \subseteq X$ such that $\bar{L}_\lambda(A) = 1$ and $\lim_{\substack{x \to \infty \\ x \in A}} F(x, x) = \alpha < 1$ then $L_\lambda(F) = \alpha$ and the random walk is $\bar{\lambda}$-TOA.

Theorem 5.3. Let (X, P) satisfy the hypotheses of Theorem 3.13, then the following are equivalent:

(a) the random walk is $\bar{\lambda}$-TOA;

(b) there exists $A \subseteq X$ such that $\underline{L}_\lambda(A) > 0$, there is a finite energy flow u^x from x to ∞ with non-zero input i_0 for every $x \in A$ and $\sup_{x \in A} \langle u^x, u^x \rangle < +\infty$.

The main goal is now to characterize the family of (inhomogeneous) graphs with absence of spontaneous breaking of continuous symmetries. From ref. 10 we know that this family contains the class of recurrent graphs. Hence, one has to weaken the concept of recurrence taking into account the properties of "almost all" the vertices of the graph. Unfortunately the thermodynamical classification fails on particularly "nasty" examples (Examples 3.1 and 3.12), although it is not clear whether these examples have physical meaning or not. Our classification enlarges the class of "recurrent" graphs (and the classification with $\bar{L}$ does so even more). We think that the techniques employed here to classify our examples may be useful to classify graphs with a finite number of "types of inhomogeneity" (in a sense which should be formalized, see also ref. 3). The connection between continuous symmetry breaking and recurrence on the average is an open problem which deserves future investigations, which could start from the analysis of graphs with only a few such types of inhomogeneity.

APPENDIX A: L_λ-MEASURABLE SETS ARE NOT AN ALGEBRA

Proposition A.1. Let $\{B_n\}_{n\in\mathbb{N}}$ be an increasing family of subsets whose union is X, and $\lambda_n(x)=\mathbb{1}_{B_n}(x)/|B_n|$. Then the class of L_λ-measurable subsets is not an algebra; in particular there exist A, B L_λ-measurable subsets of X such that $A\cap B$ is not L_λ-measurable.

Proof. Let us define, for every $n\in\mathbb{N}$, $m_n:=|B_n|$; let $\{A_k, C_k\}_{k\in\mathbb{N}}$ be a family of subsets of X such that for every $k\in\mathbb{N}$, $\{A_k, C_k\}$ is a partition of $S_k=B_k\setminus B_{k-1}$ with the following two properties

$$|A_k|-|C_k|\in\{0,\pm 1\},\qquad \forall k\in\mathbb{N},$$

$$\left|\bigcup_{i=0}^{k} A_i\right|-\left|\bigcup_{i=0}^{k} C_i\right|\in\{0,\pm 1\},\qquad \forall k\in\mathbb{N}.$$

Let us define for every $k\in\mathbb{N}$, $a_k:=|\bigcup_{i=0}^k A_i|$, $c_k:=|\bigcup_{i=0}^k C_i|$; since X is infinite, we can choose an increasing sequence of natural numbers $\{k_n\}_n$ such that $m_{k_{n+1}}/m_{k_n}\geqslant 4$. It is easy to note that by our hypotheses, for every $n, i\in\mathbb{N}$,

$$\begin{gathered}\frac{1}{2}-\frac{1}{m_n}\leqslant\frac{a_n}{m_n}\leqslant\frac{1}{2}+\frac{1}{m_n}\\ \frac{1}{2}-\frac{1}{m_n}\leqslant\frac{c_n}{m_n}\leqslant\frac{1}{2}+\frac{1}{m_n}\\ \frac{m_i-2}{m_n+2}\leqslant\frac{a_i}{a_n}\leqslant\frac{m_i+2}{m_n-2}.\end{gathered}\tag{10}$$

We finally define the two sets

$$A:=\bigcup_{i=0}^{\infty} A_i,\qquad B:=\bigcup_{i=0}^{\infty}\left(\bigcup_{j=k_{2i}+1}^{k_{2i+1}} A_j\cup\bigcup_{j=k_{2i+1}+1}^{k_{2i+2}} C_j\right);$$

by Eq. (10) (since $|A \cap B_n| = a_n$) we have that A is measurable and $L_\lambda(A) = 1/2$; similarly $||B \cap B_n| - c_n| \leqslant 1 + |\{i \in \mathbb{N} : k_i < n\}|$. Since $m_{k_{n+1}}/m_{k_n} \geqslant 4$ we have that $\lim_{n\to+\infty} |\{i \in \mathbb{N} : k_i < n\}|/m_n = 0$ (observe that if $|\{i \in \mathbb{N} : k_i < n\}| = j$ then $m_n \geqslant 4^j$), then by Eq. (10) we obtain that B is also measurable and $L_\lambda(B) = 1/2$. Moreover $A \cap B = \bigcup_{i=0}^{\infty} \bigcup_{j=k_{2i}+1}^{k_{2i+1}} A_j$, hence if n is odd

$$\frac{|A \cap B \cap B_{k_n}|}{|B_{k_n}|} \geqslant \frac{a_{k_n}}{m_{k_n}} \left(1 - \frac{a_{k_{n-1}}}{a_{k_n}}\right) \xrightarrow{n\to+\infty} \frac{1}{4};$$

similarly when n is even (since $m_{k_{n+1}}/m_{k_n} \geqslant 4$ and using Eq. (10))

$$\frac{|A \cap B \cap B_{k_n}|}{|B_{k_n}|} \leqslant \frac{a_{k_{n-1}}}{m_{k_n}}.$$

Since $\liminf_{n\to+\infty} \frac{a_{k_{n-1}}}{m_{k_{n-1}}} \frac{m_{k_{n-1}}}{m_{k_n}} \leqslant 1/8$, we have that $\underline{L}_\lambda(A \cap B) \leqslant 1/8$ meanwhile $1/4 \leqslant \bar{L}_\lambda(A \cap B)$ which implies that $A \cap B$ is not measurable. ∎

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REFERENCES

1. M. Barlow, T. Coulhon, and A. Grigor'yan, Manifolds and graphs with slow heat kernel decay, *Invent. Math.* **144**:609–649 (2001).
2. R. Burioni and D. Cassi, Universal properties of spectral dimension, *Phys. Rev. Lett.* **76**:1091–1093 (1996).
3. R. Burioni, D. Cassi, and C. Destri, Spectral partitions on infinite graphs, *J. Phys. A* **33**:3627, 3636 (2000).
4. R. Burioni, D. Cassi, and A. Vezzani, The type-problem on the average for random walks on graphs, *Eur. Phys. J. B* **15**:665 (2000).
5. D. Cassi, Local vs. average behavior on inhomogeneous structures: Recurrence on the average and a further extension of Mermin–Wagner theorem on graphs, *Phys. Rev. Lett.* **76**:2941 (1996).
6. D. Cassi and L. Fabbian, The spherical model on graphs, *J. Phys. A* **32**:L93 (1999).
7. T. Coulhon and A. Grigor'yan, Random walks on graphs with regular volume growth, *Geom. Funct. Anal.* **8**:656–701 (1998).
8. A. Grigor'yan and A. Telcs, Harnack inequalities and sub-Gaussian estimates for random walks, *Math. Ann.* **324**:521–556 (2002).
9. A. Grigor'yan and A. Telcs, Sub-Gaussian estimates of heat kernels on infinite graphs, *Duke Math. J.* **109**:451–510 (2002).

10. F. Merkl and H. Wagner, Recurrent random walks and the absence of continuous symmetry breaking on graphs, *J. Stat. Phys.* **75**:153–165 (1994).
11. E. Hille, *Analytic Function Theory*, Vol. I (Chelsea Publ. Co., New York, 1959).
12. J. F. C. Kingman, The exponential decay of Markov transition probabilities, *Proc. London Math. Soc.* **13**:337–358 (1963).
13. E. Kreyszig, *Introductory Functional Analysis with Applications* (Wiley, New York, 1978).
14. T. Lyons, A simple criterion for transience of a reversible Markov chain, *Ann. Probab.* **11**:393–402 (1983).
15. G. Pólya, Uber eine Aufgabe der Wahrscheinlichkeitstheorie betreffend die Irrfahrt im Straßennetz, *Math. Ann.* **84**:149–160 (1921).
16. W. Rudin, *Principles of Mathematical Analysis* (McGraw–Hill, 1953).
17. A. Telcs, Isoperimetric inequalities for random walks, *Potential Anal.* **19**:237–249 (2003).
18. W. Woess, *Random Walks on Infinite Graphs and Groups*, Cambridge Tracts in Mathematics, Vol. 138 (Cambridge University Press, 2000).
19. M. Yamasaki, Discrete potentials on an infinite network, *Mem. Fac. Sci., Shimane Univ.* **13**:31–44 (1979).
20. F. Zucca, Mean value properties for harmonic functions on graphs and trees, *Ann. Mat. Pura Appl. (4)* **181**:105–130 (2002).